Construction of surfaces of general type with p_g=0 via Q-Gorenstein smoothings

Construction of surfaces of general type with $p_g = 0$ via $\mathbb{Q}$-Gorenstein smoothings

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For nonsingular projective curve C, genus C=0 implies C is isomorphic to the projective line $\mathbb{P}^1$.

The projective plane $\mathbb{P}^2$ has $p_g(\mathbb{P}^2) = 0$ and $q(\mathbb{P}^2) = 0$.
so naturally

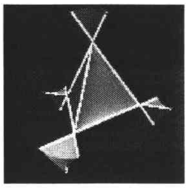
Question (by Max Noether)
Is every nonsingular projective surface S with $p_g(S) = q(S) = 0$ a rational surface?

$p_g(S) := h^0(K_S)$ geometric genus of S, K_S canonical divisor
$q(S) := h^0(\Omega_S)$ irregularity of S

Answer: There are counterexamples:

- Enriques surfaces (Enriques, 1894): normalization of singular sextic in $\mathbb{P}^3$, $\kappa = 0$.

$$(x_0x_1x_2)^2 + (x_0x_1x_3)^2 + (x_0x_2x_3)^2 + (x_1x_2x_3)^2$$
$$+x_0x_1x_2x_3(x_0^2 + x_1^2 + x_2^2 + x_3^2) = 0 \quad \text{in } \mathbb{P}^3.$$



- Rationality criterion was proved by Castelnouvo:

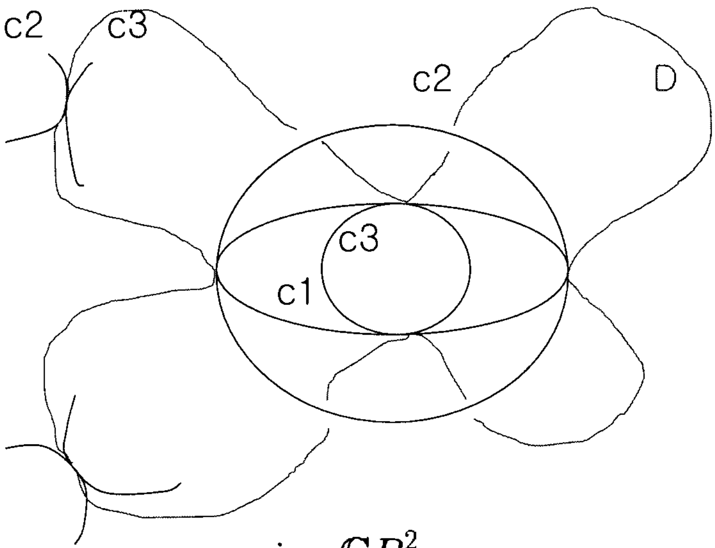
$$P_2(X) = h^0(X, 2K_X) = 0, q(X) = 0$$

$\Rightarrow$ X is a rational surface.

- Godeaux surface (Godeaux, 1931): minimal surface of general type with $p_g = 0, K^2 = 1$ obtained as the quotient of a smooth quintic surface in $\mathbb{P}^3$ by a free $\mathbb{Z}_5$ -action.

- The first example of a surface general type with $p_g = 0, K^2 = 2$ was constructed by Campedelli in the 30's as a ramified double cover of $\mathbb{P}^2$;
  more precisely as the desingularization of a double cover of $\mathbb{P}^2$ branched along a reducible curve of degree 10 with 6 [3, 3] points not lying on a conic.
  Nowdays minimal surface of general type with $p_g = 0, K^2 = 2$ are called (numerical) Campedelli surfaces.

c1, c2, c3 conics, D quartic



in $\mathbb{C}P^2$

Severi conjectured (1949)

$p_g(X) = 0, H_1(X, \mathbb{Z}) = 0 \;\; \Rightarrow \;\; X$ rational surface ?

Dolgachev constructed elliptic surfaces with

$p_g = 0, \pi_1 = 1, K^2 = 0, \;\; \kappa = 1.$

Nowdays these surfaces are called Dolgachev surfaces.

Question

Is there a minimal surface of general type with

$p_g(X) = 0, \pi_1(X) = 1 \, (or \; H_1(X, \mathbb{Z}) = 0)?$

Surfaces of general type X with $p_g = 0$ (and so $q = 0$) in principle can be classified, since the moduli space has finitely many components by Giesker's theorem.
By the Miyaoka-Yau inequality,
X minimal $\Rightarrow$ $1 \leq K_X^2 \leq 9$.

In practice not much is known. Surfaces of general type with geometric genus zero have been studied by algebraic geometers for a long time and plenty of examples have been constructed, but at present a classification seems still out of reach.

Surfaces of general type with $p_g = 0$ are important to classify surfaces of general type, and to study threefolds with a fibration to a curve.

How does one construct examples?
Mainly two appoaches (classical methods)

- Godeaux - taking quotients by group actions of known surfaces (finite quotient methods)
- Campedelli – constructing suitable covers of known surfaces (covering methods)

Barlow [Invent. Math. 1985] constructed a simply connected minimal surface of general type with $p_g = 0, K^2 = 1$ obtained by a variation of the Godeaux construction, in which the group has some isolated fixed points.

It was the first and up to 2006 was the only known example of a simply connected surface of general type with vanishing geometric genus .

Recently, Bauer, Catanese, and others construct many examples of surfaces of general type with $p_g = 0(\pi_1 \neq 1)$ and gave a classification that admit an unramified covering which is isomorphic to a product of curves.

(generalization Beauville's construction)

It is the first systematic way to find many examples of surfaces of general type with $p_g = 0$.

A interesting and hard question concerning these surfaces is the construction of simply connected examples, which are of great interest also in the study of differentiable four-manifolds.

X topological 4-manifold

$Q : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z})$

(unimodular symmetric bilinear form)

Freedman's theorem: If $\pi_1(X) = 1$ then X is uniquely determined up to homeomorphism by Q.

A simply connected surface of general type with $p_g = 0$ is homeomorphic (not diffeomorphic) to a rational surface.

Barlow surface is homeomorphic to $\mathbb{P}^2\#8\bar{\mathbb{P}}^2$.

All known methods seem to be not useful in producing new simply connected examples and it has long been an open question whether there exist simply connected surfaces of general type with $p_g = 0, K^2 > 1$.

Y.Lee, J. Park [Invent. Math. 2007] constructed a minimal surface of general type with $p_g = 0, K^2 = 2, \pi_1 = 1$ by using a new method.

- Idea from a moduli space

Assume that there is a surface of general type X satisfying the given numerical invariants $\chi(\mathcal{O}_X) = p_g - q + 1, K^2$.

Gieseker proved that there is a quasi-projective moduli space $\mathcal{M}$ of X by Geometric Invariant Theory.

$$\dim \mathcal{M} \geq h^1(X, T_X) - h^2(X, T_X)$$

Compactify $\mathcal{M}$ by adding points corresponding to singular surfaces at boundary. There is a natural way to do this using Minimal Model Program of threefolds.

- Idea from Park's symplectic construction [Invent, 2005]

- Examples are constructed by a new method ($\mathbb{Q}$-Gorenstein smoothings of singular rational surfaces).

- The main example construction goes as follows:

Step 1: choose a special pencil of cubics in $\mathbb{P}^2$

$$\lambda(3line) + \mu(conic + line)$$

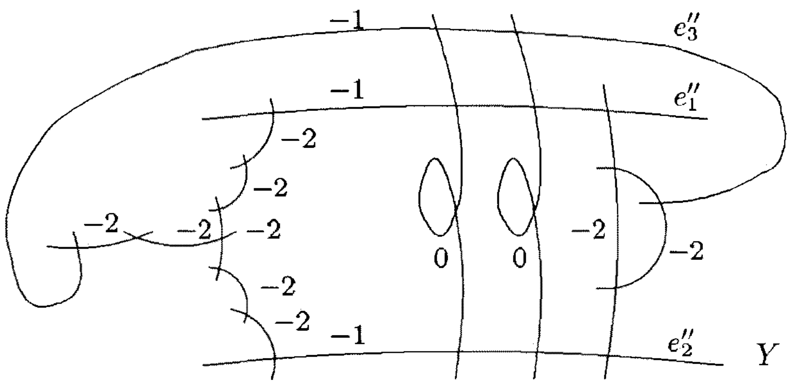
blows up its base locus —> a elliptic rational surface

Step 2: blows up further

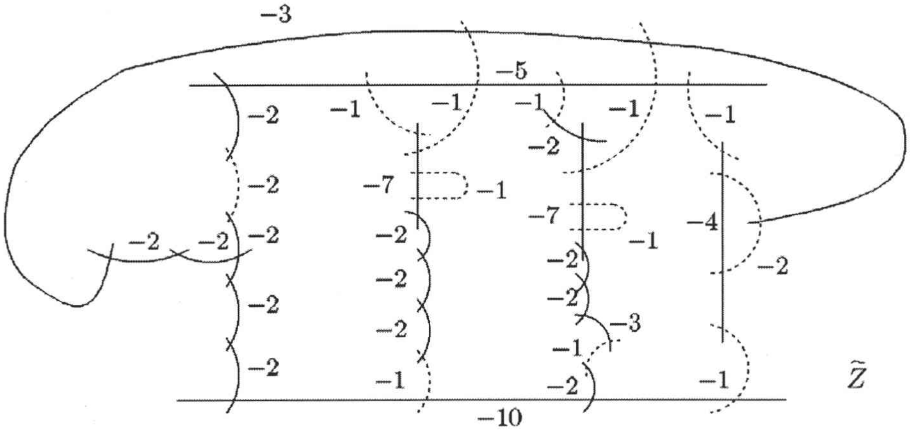
—> 5 disjoints chains of rational curves

—> blown down them to get a singular rational surface X.

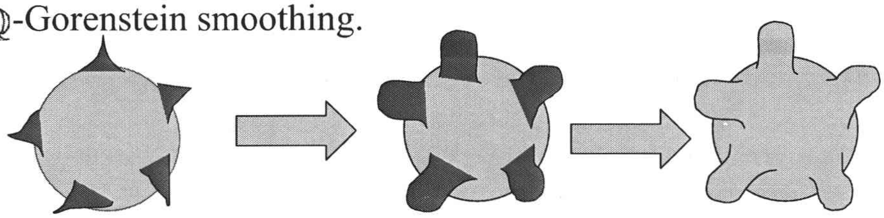
Step 1



Step 2



Every singularity of X is of class T, namely it admits a local $\mathbb{Q}$-Gorenstein smoothing.



Step 3: Using deformation theory, there is indeed a global $\mathbb{Q}$ -Gorenstein smoothing of X,

a one parameter family $\mathcal{X} \to \Delta$ of projective surfaces s.t.
the central fiber is X ;
the general fibre X_t is smooth and projective;
the relative canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$ -Cartier.

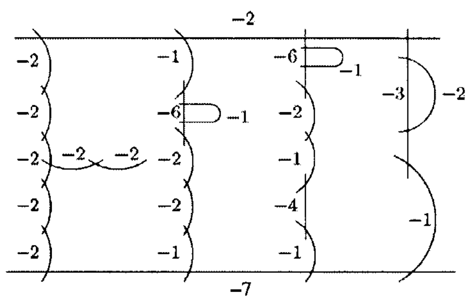
X_0(five singularities of class T) $\rightsquigarrow$ (deformation to) X_t

Step 4:

What properties does X_t have?

- By deformation, $K^2_{X_t} = 2$, $p_g(X_t) = q(X_t) = 0$.
- By configuration of the construction, X_t is minimal.
- By configuration of the construction and by using standard argument of Minor fiber, $\pi_1(X_t) = 1$

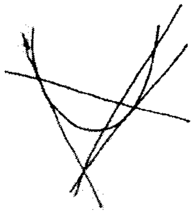
Example. $(p_g = 0, K^2 = 1, \pi_1 = 1)$

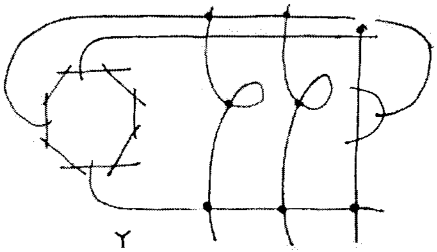


(-7) – (-2) – (-2) – (-2), (-6) – (-2) – (-2),
(-2) – (-6) – (-2) – (-3), (-4)

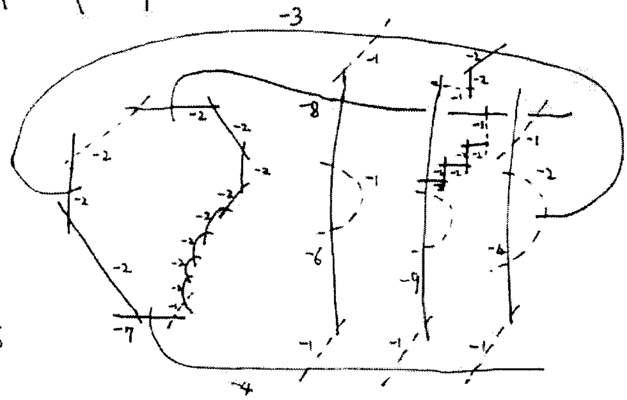
K^2 = -11+12=1

$K^2 = 3$, $p_g = 0$, $\pi_1 = 1$ (H. Park, J. Park, D. Shim)





Y



$Y \# 21\overline{\mathbb{P}}^2$

$C_{2,1}$: -4

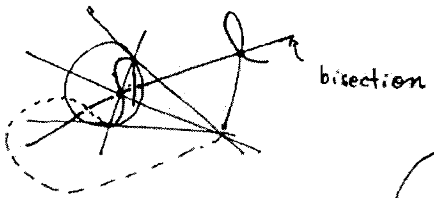
$C_{7,1}$: -9 — -2 — -2 — -2 — -2 — -2

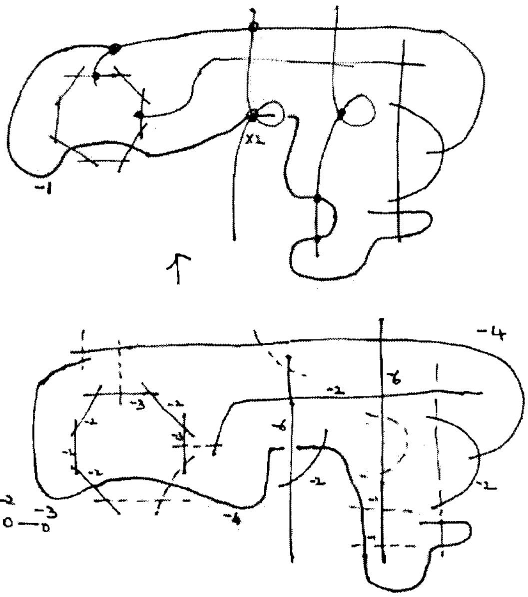
$C_{19,5}$: -4 — -7 — -2 — -2 — -3 — -2 — -2

$C_{35,6}$: -6 — -8 — -2 — -2 — -2 — -3 — -2 — -2 — -2 — -2

$K^2 = -21 + 24 = 3$

$K^2 = 4$, $p_g = 0$, $\pi_1 = 1$ (H. Park, J. Park. D. Shim)





$Y \# 9\overline{\mathbb{P}}^2$

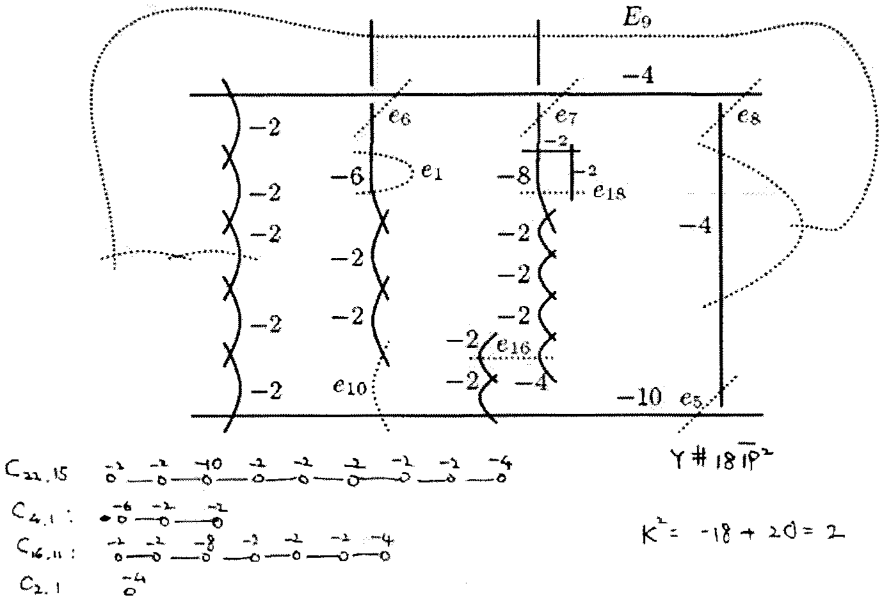
$C_{252,145}$:

-2 — -4 — -6 — -2 — -6 — -2 — -4 — -2 — -2 — -2 — -3 — -2 — -3

$K^2 = -9 + 13 = 4$

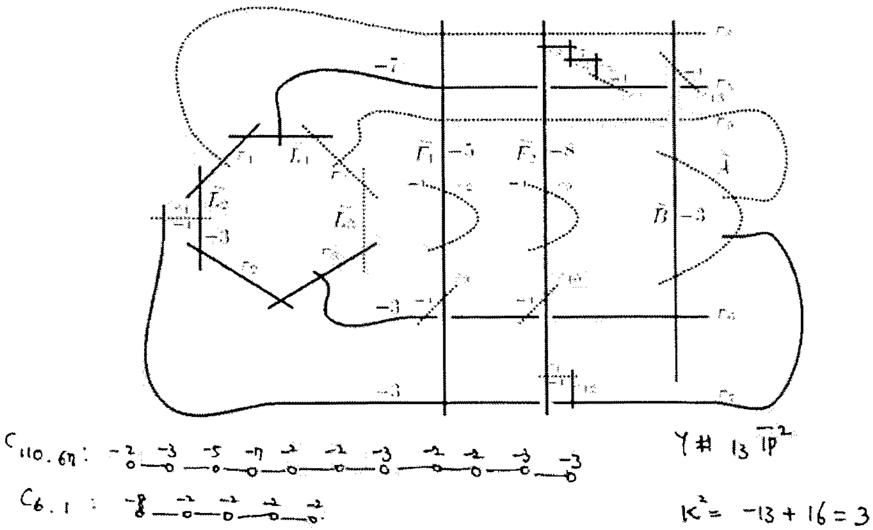
Example $(p_g = 0, K^2 = 2, H_1(X, \mathbb{Z}) = \mathbb{Z}_2)$

Lee-Park

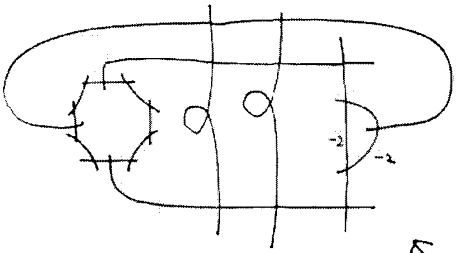


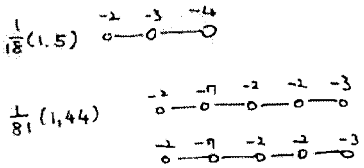
Example $(p_g = 0, K^2 = 3, H_1(X, \mathbb{Z}) = \mathbb{Z}_2)$

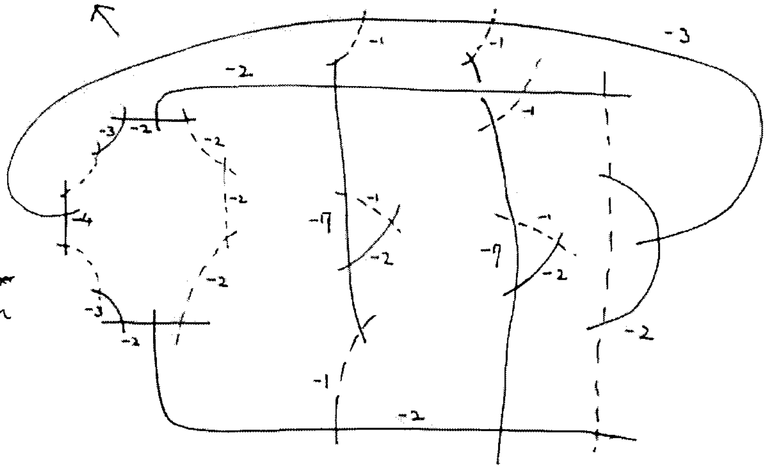
H.Park - J. Park - D. Shin



$K^2=2$, $p_g=0$, $H_1=\mathbb{Z}_3$ (Lee - Park)







$K^2 = -10 + 13 - 1 = 2$

-10: # of blow-ups

$13 = 3+5+5$

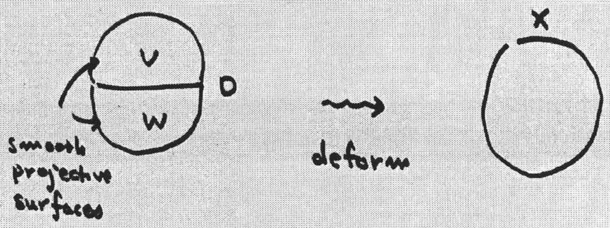
1: 2nd betti number of Milnor fiber

The construction problem,
to find a simply connected surface of general type
with $p_g = 0$ and given $1 \leq K^2 \leq 4$, is solved.

Question: Is there a minimal simply connected surface of general type with $p_g = 0$ and $5 \leq K^2 \leq 8$?

Remark: Similar construction does not work for $5 \leq K^2 \leq 8$.
In all constructions, $H^2(X_t, T_{X_t}) = 0$.
$H^1(X_t, T_{X_t}) = 10 - 2K^2_{X_t}$.

Symplectic construction should be modified.



$$N_{D|V} \otimes N_{D|W} = \mathcal{O}_D$$

$\Rightarrow$ $\exists$ symplectic 4-manifold X

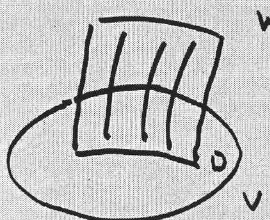
complex structure on X ?

need some complicated conditions

V smooth projective surface

D smooth projective curve in V

Assume $\eta \neq 0 \in H^1(N_{D|V}) \Rightarrow 0 \to N_{D|V} \to \mathcal{E} \to \mathcal{O}_D \to 0$ (rank 2 vector bundle $\mathcal{E}$)



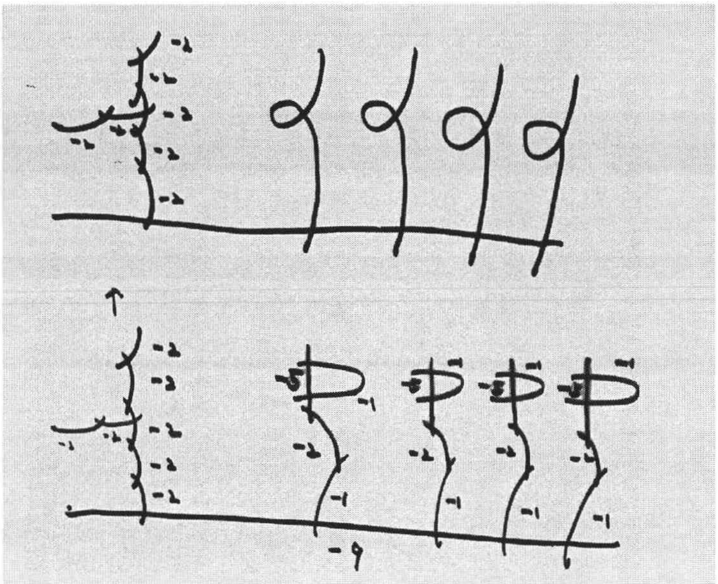
$\Rightarrow$ Let $W = \mathbb{P}_D(\mathcal{E})$

$$N_{D|V} \otimes N_{D|W} = \mathcal{O}_D$$

But no complex structure on X

if η is not image $H^1(T_V) \to H^1(N_{D|V})$

Modified version of Park's symplectic construction.



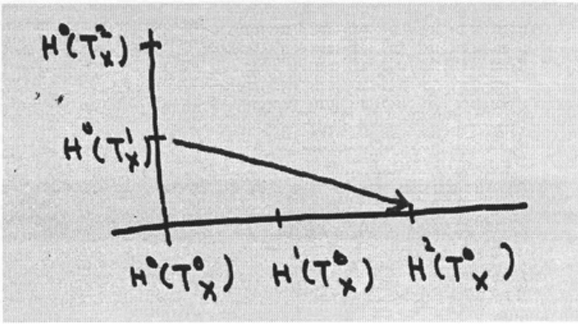
$\mathbb{Q}$-Gorenstein deformations

$(X_0, 0)$ germ of two-dimension quotient singularity
1st order deformation (local) $\leftrightarrow$ $T^1_{X_0} = Ext^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$
Obstruction space lies in $T^2_{X_0} = Ext^2_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$

X normal projective surface with quotient singularities
1st order deformation (global) $\leftrightarrow$ $\mathbf{T}^1_X = \mathbf{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$
Obstruction space lies in $\mathbf{T}^2_X = \mathbf{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$

Spectral sequence $E_2^{p,q} = H^p(X, T^q_X) \Rightarrow \mathbf{T}^{p,q}_X$

$$H^i(X, T^j_X) = 0 \text{ if } i, j \geq 1$$



$$0 \to H^1(T^0_X) \to \mathbf{T}^1_X \to H^0(T^1_X) \to H^2(T^0_X) \to T^2_0(X) \to 0$$

$$0 \to T^2_0(X) \to \mathbf{T}^2_X \to H^0(T^2_X) \to 0$$

Key Part: $\mathrm{Ker}[H^0(T^1_X) \to H^2(T^0_X)]$

[Wahl], [Manetti] If $H^2(T^0_X) = 0$ then every local deformation $(X_0, 0)$ of the singularities may be globalized.
What condition implies $H^2(T^0_X) = 0$?

Main two technical Lemmas
1. Let V be the minimal resolution of X and E be the reduced Exceptional divisors.

$$H^2(T_V(-log\, E)) = 0 \Rightarrow H^2(T^0_X) = 0$$

2. From the constructions, our examples satisfy

$$H^2(T_V(-log\, E)) = 0.$$

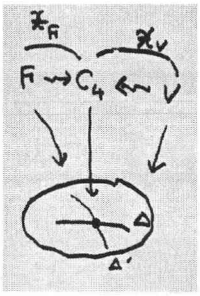
Consider special quotient singularities (singularity of class T)

1) To have a nice smoothing part in the 1st order deformation which has no obstruction (local smoothing),
2) To control numerical invariants and to use topological properties (Milnor fiber) of a general fiber of smoothing .

Example) $V \subset \mathbb{P}^5$ Veronese surface $\quad C_V \subset \mathbb{P}^6$ cone of V

$F = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2)) \subset \mathbb{P}^5 \quad C_F \subset \mathbb{P}^6$ cone of F

$C_4 \subset \mathbb{P}^5$ cone of the rational normal curve of degree 4



$$K_{C_4}^2 = 9 = K_V^2, \; K_F^2 = 8$$

$K_{\mathcal{X}_V/\Delta}$ $\mathbb{Q}$-Cartier, $\quad K_{\mathcal{X}_F/\Delta'}$ not $\mathbb{Q}$ -Cartier

If X_0 is a quotient singularity of type $\frac{1}{dr^2}(1, dra-1)$.
then $X_0 = Y_0/ < \sigma >, \; Y_0 : xy - z^{dr} = 0.$
$\sigma : (x, y, z) \to (\xi x, \xi^{-1} y, \xi^a z)$ $\quad \xi$ is a primitive r-th root of unity

And there is a $\mathbb{Q}$-Gorenstein smoothing.

$$X = Y/ < \sigma > \to \Delta, \; Y : xy - z^{dr} + t = 0$$

σ acts on Y via $\quad \sigma : (x, y, z, t) \to (\xi x, \xi^{-1} y, \xi^a z, t)$

K_Y Cartier $\quad => \; K_X$ $\mathbb{Q}$-Cartier.

There is a d-dimensional $\mathbb{Q}$-Gorenstein smoothing.
Conversely, if there is a $\mathbb{Q}$-Gorenstein smoothing then it is a RDP or a cyclic quotient singularity of type $\frac{1}{dr^2}(1, dra-1)$.

[Kollár, Shepherd-Barron], [Wahl]

Let $(X_0, 0)$ be a two-dimensional quotient singularity. If $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then either $(X_0, 0)$ is a RDP or it is a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$ where $\gcd(n, a) = 1$.

Moreover, every such cyclic quotient singularity admits a $\mathbb{Q}$-Gorenstein smoothing.

i) $\overset{-4}{\bullet}$ $\quad$ $\overset{-3}{\bullet}-\overset{-2}{\bullet}-\overset{-2}{\bullet}\cdots\overset{-2}{\bullet}-\overset{-3}{\bullet}$ class T

ii) If $\overset{-b_1}{\bullet}-\overset{-b_2}{\bullet}-\cdots-\overset{-b_{r-1}}{\bullet}-\overset{-b_r}{\bullet}$ class T

then $\overset{-2}{\bullet}-\overset{-b_1}{\bullet}-\overset{-b_2}{\bullet}-\cdots-\overset{-b_{r-1}}{\bullet}-\overset{-b_r-1}{\bullet}$

$\overset{-b_1-1}{\bullet}-\overset{-b_2}{\bullet}-\cdots-\overset{-b_r}{\bullet}-\overset{-2}{\bullet}$ class T

iii) Every singularity of class T that is not RDP can be obtained by starting with one of (i) and iterating to steps in (ii)

e.g. $\overset{-4}{\bullet}$ $\quad$ $\overset{-5}{\bullet}-\overset{-2}{\bullet}$

$\overset{-6}{\bullet}-\overset{-2}{\bullet}-\overset{-2}{\bullet}$ $\quad$ $\overset{-2}{\bullet}-\overset{-5}{\bullet}-\overset{-3}{\bullet}$ $\quad \dots$

Main Lemma 1

Let V be the minimal resolution of X and E be the reduced exceptional divisors.

Then $\quad H^2(T_V(-log\, E)) = 0 \Rightarrow H^2(T_X^0) = 0.$

Idea (suggested by Manetti)

Let $\pi : V \to X$. Then $\pi_* T_V = T_X^0$ [Burns-Wahl].

$0 \to \pi_* T_V(-log\, E) \to T_X^0 \to \Delta \to 0$, Δ supported in the Sing X.

$$H^2(T_X^0) = H^2(\pi_* T_V(-log\, E)).$$

$$R^1\pi_* T_V(-log\, E) = R^2\pi_* T_V(-log\, E) = 0 :$$

We may assume that X is affine. $0 \to T_V(-Z) \to T_V(-log\, E) \to T_Z^0 \to 0$

Z is effective divisor supported in E [Burns-Wahl].

Z sufficiently big ($-Z$ is π-ample) $H^i(T_V(-Z)) = 0, i > 0.$

$H^1(T_V(-log\, E)) = H^1(T_Z^0)$, $H^2(T_V(-log\, E)) = 0.$

$H^1(T_Z^0) = 0$ [Laufer] Two dimensional quotient singularity is taut.

Main Lemma 2

From the constructions, our examples satisfy

$$H^2(T_{\tilde{Z}}(-log\, E)) = 0.$$

Key Lemma 1: V nonsingular surface, D s.n.c. divisor in V.

$f : V' \to V$ blow-up of V at a point in D. Set $D' = f^{-1}(D)_{red}$.

Then $h^2(T_{V'}(-log\, D')) = h^2(T_V(-log\, D))$.

Key Lemma 2: Let Z be a blow-up at two singular points of two nodal curves in special fibers.

It is enough to prove that

$H^2(Z, T_Z(-log\, D_Z)) = 0.$ $D_Z = F_1 + F_2 + F + D + S_1 + S_2 + S_3.$

F_i (-4)-curve in special fibers, F proper transform of conic,

D $\tilde{E}_6$- one (-2)-curve, S_i section

Key Lemma 3: It is enough to prove $H^0(Z, \Omega_Z(K_Z + F_1 + F_2 + F + D)) = 0.$

$$H^0(Z, \Omega_Z(K_Z + F_1 + F_2 + F + D)) = 0 \subseteq H^0(Y, \Omega_Y(C + F + D))$$

C general fiber

$$H^0(Y, \Omega_Y(C + F + D)) = H^0(Y, \Omega_Y(3C - E - D'))$$

E line, $D + D' = \tilde{E}_6$ fiber

Key Lemma 4: Y rational elliptic surface. Assume that the elliptic fibration $g : Y \to \mathbb{P}^1$ is relatively minimal without multiple fibers.

C general fiber of $g : Y \to \mathbb{P}^1$. Then

$$H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(k)) = H^0(Y, \Omega_Y(kC)), k \geq 1.$$

Thank you for listening.